

FUJITA'S CONJECTURE AND FROBENIUS AMPLITUDE

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ABSTRACT. We prove a version of Fujita's Conjecture in arbitrary characteristic, generalizing results of K.E. Smith. Our methods use the Frobenius morphism, but avoid tight closure theory. We also obtain versions of Fujita's Conjecture for coherent sheaves with certain ampleness properties.

1. INTRODUCTION

Fujita's Conjecture is a deceptively simple open question in classical algebraic geometry. Given a smooth complex projective variety X of dimension d and an ample line bundle \mathcal{L} , the conjecture predicts that

- (1) The line bundle $\omega_X \otimes \mathcal{L}^m$ is generated by global sections for $m \geq d + 1$.
- (2) The line bundle $\omega_X \otimes \mathcal{L}^m$ is very ample for $m \geq d + 2$.

While the conjecture was stated in some form more than two decades ago [F], thus far the global generation conjecture has only been proven for $\dim X \leq 4$ [EL, Ka]. Also, over \mathbb{C} , many other "Fujita Conjecture type" theorems have been proven. We direct the reader to [L, Section 10.4] for a partial summary.

Since these proofs rely on the Kodaira Vanishing Theorem and its generalizations, it was surprising when K.E. Smith proved in arbitrary characteristic, via tight closure theory, that if \mathcal{L} is ample and *generated by global sections*, then $\omega_X \otimes \mathcal{L}^{d+1}$ is generated by global sections [S1]. This result was recovered in [Hara], via a characteristic p analogue of multiplier ideals. Using tight closure methods, Smith also proved that if \mathcal{L} is very ample and $(X, \mathcal{L}) \neq (\mathbb{P}^n, \mathcal{O}(1))$, then $\omega_X \otimes \mathcal{L}^d$ is generated by global sections [S2]. (Note that Smith and Hara allowed F -rational singularities, but we will remain in the smooth case.)

Using characteristic p methods, but staying within the realm of algebraic geometry, we will prove

Theorem 1.1. *Let X be a projective scheme of pure dimension d , smooth over a field k of arbitrary characteristic. Let \mathcal{L} be an ample, globally generated line bundle and let \mathcal{H} be an ample line bundle. Then*

- (1) $\omega_X \otimes \mathcal{L}^d \otimes \mathcal{H}$ is generated by global sections.
- (2) $\omega_X \otimes \mathcal{L}^{d+1} \otimes \mathcal{H}$ is very ample.

As far as the author knows, this is the greatest generality in which Fujita's Conjecture has been proven in characteristic p . Also, this is the first characteristic p version of the very ampleness part of the Conjecture. On the other hand, the theorem above can be deduced easily over \mathbb{C} , using the Kodaira Vanishing Theorem and Castelnuovo-Mumford regularity [L, Example 1.8.23]. While we still use

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regularity, we of course avoid Kodaira Vanishing. Thus we hope our method will be useful in other situations where one must avoid Kodaira Vanishing. (For those wishing to avoid vanishing theorems altogether, one may prove Theorem 1.1 as in [L, Remark 10.4.5] when $\mathcal{H} = \mathcal{L}$ is very ample, regardless of $\text{char } k$.)

In [Ar], Arapura undertakes a thorough investigation of so-called F -ample coherent sheaves and generalizes the definition to characteristic 0 (see Definition 3.1). Our method easily generalizes Theorem 1.1(1) to the case where \mathcal{H} is any F -ample coherent sheaf (see Theorem 3.4).

With more work, in Theorem 4.2, we generalize to the case where \mathcal{H} is replaced with an F -ample coherent sheaf tensored with a p -ample coherent sheaf. (See Definition 4.1 for the definition of p -ample.) This allows us to prove

Theorem 1.2. *Let X be a projective scheme of pure dimension d , smooth over a field k . Let \mathcal{F}_n be a sequence of coherent sheaves. Then the following are equivalent:*

- (1) *For any coherent \mathcal{G} , there exists n_0 such that $\mathcal{G} \otimes \mathcal{F}_n$ is generated by global sections for $n \geq n_0$.*
- (2) *For any coherent locally free \mathcal{E} , there exists n_1 such that $\mathcal{E} \otimes \mathcal{F}_n$ is generated by global sections for $n \geq n_1$.*
- (3) *For any invertible sheaf \mathcal{H} , there exists n_2 such that $\mathcal{H} \otimes \mathcal{F}_n$ is p -ample for $n \geq n_2$.*

This answers [Ke, Question 7.5], at least when X is smooth.

2. REDUCTIONS

In this section, we verify that Theorems 1.1, 3.4, and 4.2 can be reduced to the case of k algebraically closed and $\text{char } k = p > 0$. The material is standard, yet somewhat scattered in the literature. The familiar reader may wish to move to the next section. Thus the definitions of F -ample and p -ample will wait until Definitions 3.1 and 4.1.

First, we must check that we can extend our field k .

Lemma 2.1. *Let X be a projective scheme of pure dimension d , smooth over a field k . Let $k \rightarrow k'$ be a field extension. Let $X' = X \times_k k'$ and $p : X' \rightarrow X$ be the projection. Then*

- (1) *X' is of pure dimension d and smooth over k' ,*
- (2) *$\omega_{X'/k'} \cong p^* \omega_{X/k}$,*
- (3) *a line bundle \mathcal{L} on X is ample (resp. very ample) if and only if $p^* \mathcal{L}$ is ample (resp. very ample),*
- (4) *a coherent sheaf \mathcal{F} is generated by global sections (resp. F -ample, p -ample) if and only if $p^* \mathcal{F}$ is generated by global sections (resp. F -ample, p -ample).*

Proof. Since X is smooth over k and of pure dimension d , we have that $\Omega_{X/k}$ is locally free of rank d [AK, VII.5.1]. Now $p^* \Omega_{X/k} \cong \Omega_{X'/k'}$ [AK, p. 110]. So $\Omega_{X'/k'}$ is also locally free of rank d and we must have that X' is smooth and of pure dimension d [AK, VII.5.3]. We also immediately have (2).

The claim (3) is [EGA, IV₂, 2.7.2].

If $\oplus \mathcal{O}_X \rightarrow \mathcal{F}$ is a surjection, then $\oplus p^* \mathcal{O}_X \cong \oplus \mathcal{O}_{X'} \rightarrow p^* \mathcal{F}$ is a surjection. On the other hand, since $p : X' \rightarrow X$ is faithfully flat, if $H^0(\mathcal{F}) \otimes_k \mathcal{O}_X \rightarrow \mathcal{F}$ has non-zero cokernel, then there is still a non-zero cokernel upon pulling back. This proves the claim about globally generated sheaves.

If $\text{char } k = 0$, then the claim regarding F -ampleness or p -ampleness is clear by definition.

Consider the case of $\text{char } k = p > 0$. The absolute Frobenius morphism commutes with any morphism of ringed spaces. Thus, $p^*(\mathcal{F}^{(p^n)}) = (p^*\mathcal{F})^{(p^n)}$. So the claim regarding F -ampleness follows from [Ke, Lemma 3.8].

To show a sheaf \mathcal{F} is p -ample, it is sufficient to show that for a fixed very ample sheaf $\mathcal{O}_X(1)$, we have that for any $b \in \mathbb{N}$, $\mathcal{O}_X(-b) \otimes \mathcal{F}^{(p^n)}$ is generated by global sections for $n \gg 0$. But this happens if and only if $p^*\mathcal{O}_X(-b) \otimes p^*\mathcal{F}^{(p^n)}$ is generated by global sections. Given (3), we have the claim about p -ampleness. \square

We now consider the case of $\text{char } k = 0$. Our main tool is arithmetic thickening. That is, given $X \rightarrow \text{Spec } k$, we can find a sub-algebra A of k , finitely generated over \mathbb{Z} , and a scheme \tilde{X} with flat morphism to $\text{Spec } A$ such that $\tilde{X} \times_A k = X$. By shrinking A via localization, any finite diagram of coherent sheaves on X can be translated to a diagram of coherent sheaves on \tilde{X} . We refer the reader to [Ar, §1] for more on thickenings.

Since A is finitely generated over \mathbb{Z} , the residue fields at the closed points of $\text{Spec } A$ are finite fields, hence perfect of characteristic $p > 0$. By working on closed fibers, results on the generic fiber and hence X can often be deduced.

Lemma 2.2. *Let X be a projective scheme of pure dimension d , smooth over a field k with $\text{char } k = 0$. Let \mathcal{L} be an ample line bundle, and let \mathcal{F} be a globally generated coherent sheaf. Then there exists an arithmetic thickening $f : \tilde{X} \rightarrow S = \text{Spec } A, \tilde{\mathcal{L}}, \tilde{\mathcal{F}}$ such that at every (closed) fiber \tilde{X}_s of f*

- (1) $\tilde{\mathcal{L}}_s$ is ample,
- (2) $\tilde{\mathcal{F}}_s$ is globally generated,
- (3) the morphism f is smooth,
- (4) each X_s is a smooth scheme of pure dimension d ,
- (5) $\omega_{\tilde{X}/A} \otimes_A k = \omega_X$,
- (6) $\omega_{\tilde{X}/A} \otimes_A k(s) = \omega_{X_s}$.

Proof. Let $k(\nu)$ be the residue field at the generic point of S . Given Lemma 2.1, we may replace k with $k(\nu)$ and X with $X \times_A k(\nu)$.

Now we can shrink S so that $\tilde{\mathcal{L}}$ is ample [EGA, III₁, 4.7.1]. And S can be shrunk again so there is surjection $H^0(\tilde{\mathcal{F}}_s) \otimes_A \mathcal{O}_{X_s} \rightarrow \tilde{\mathcal{F}}_s$ at each fiber X_s [EGA, IV₃, 9.4.2].

We can shrink S to make f smooth at every fiber [EGA, IV₃, 12.2.4]. Hence the morphism f is smooth over $\text{Spec } A$ [AK, VII.1.8].

Since f is smooth, the sheaf of differentials $\Omega_{\tilde{X}/A}$ is locally free [AK, VII.5.1]. Shrinking S again, we can make the rank of $\Omega_{\tilde{X}/A}$ constant [EGA, IV₂, 2.5.2], equal to $d = \dim X$. This makes each fiber a smooth scheme of pure dimension d . The claims (5) and (6) are true for $\Omega_{\tilde{X}/A}$ by the discussion on [AK, p. 110]. Hence they are also true for the $\omega_{\tilde{X}/A} = \wedge^d \Omega_{\tilde{X}/A}$. \square

Note that if a coherent sheaf \mathcal{F} over a field of characteristic 0 is F -ample or p -ample, then, by definition, \mathcal{F}_s is F -ample or p -ample on each closed fiber. Thus, all hypotheses of Theorems 1.1, 3.4, and 4.2 can be translated to the closed fibers of an arithmetic thickening.

Finally, we need that the conclusions of Theorems 1.1, 3.4, and 4.2 on a closed fiber imply the conclusion for our original scheme over a field of characteristic 0.

Lemma 2.3. *Let X be a projective scheme over a field k with $\text{char } k = 0$. Let \mathcal{L} be a line bundle and \mathcal{F} a coherent sheaf. Then there exists an arithmetic thickening $f : \tilde{X} \rightarrow S = \text{Spec } A, \tilde{\mathcal{L}}, \tilde{\mathcal{F}}$.*

- (1) *If $\tilde{\mathcal{L}}_s$ is very ample on some closed fiber X_s , then \mathcal{L} is very ample.*
- (2) *If $\tilde{\mathcal{F}}_s$ is globally generated on some closed fiber X_s , then \mathcal{F} is globally generated.*

Proof. Again we may replace k with $k(\nu)$. Since S is affine, the discussion following [EGA, III₁, 4.7.1] proves the statement regarding \mathcal{L} .

If $\oplus \mathcal{O}_{X_s} \rightarrow \mathcal{F}_s$ is surjective, then Nakayama's Lemma shows that $\oplus \mathcal{O}_{\tilde{X}} \rightarrow \tilde{\mathcal{F}}$ is surjective at the stalk at s . But then the map is surjective at the generic point as well. \square

Note that since the definition of p -ample in characteristic 0 depends on *all* closed fibers for some thickening, we cannot reduce to the characteristic p case to prove Theorem 1.2. However, we will see that the proof is quite easy given Theorem 4.2.

3. FUJITA'S CONJECTURE FOR F -AMPLE SHEAVES

In this section, we shall prove Theorem 3.4 [Ar]. We recall the definition of F -ample (also known as “cohomologically p -ample” [H2]).

Definition 3.1. Let X be a projective scheme over a field k , and let \mathcal{F} be a coherent sheaf. If $\text{char } k = p > 0$ and F is the absolute Frobenius morphism, then define $\mathcal{F}^{(p^n)} = F^{*n} \mathcal{F}$. The sheaf \mathcal{F} is F -ample if for any locally free coherent sheaf \mathcal{E} , there exists n_0 such that

$$H^q(X, \mathcal{E} \otimes \mathcal{F}^{(p^n)}) = 0, \quad q > 0, n \geq n_0.$$

If $\text{char } k = 0$, then \mathcal{F} is F -ample if and only if \mathcal{F}_s is F -ample for all closed fibers on some arithmetic thickening.

Note that if \mathcal{F} is F -ample and \mathcal{L} is an ample line bundle, then $\mathcal{F} \otimes \mathcal{L}$ is F -ample [Ar, Theorem 4.5].

We will also use *Castelnuovo-Mumford regularity* of a sheaf. Let \mathcal{L} be an ample, globally generated line bundle. Recall that a coherent sheaf \mathcal{G} is m -regular (with respect to \mathcal{L}) if

$$H^q(X, \mathcal{G} \otimes \mathcal{L}^{m-q}) = 0, \quad q > 0.$$

We only need the fact that if \mathcal{G} is m -regular, then $\mathcal{G} \otimes \mathcal{L}^m$ is generated by global sections [L, 1.8.5, 1.8.14]. While regularity is usually defined for schemes over an algebraically closed field, the reductions of Lemma 2.1 show that we may work over any field.

We are interested in the regularity of direct images under the absolute Frobenius F .

Lemma 3.2. *Let X be a projective scheme over a perfect field k of positive characteristic, and let \mathcal{L} be an ample, globally generated line bundle. For any locally free coherent sheaf \mathcal{E} and F -ample coherent sheaf \mathcal{F} , there exists n_0 such that*

$$F_*^n(\mathcal{E}) \otimes \mathcal{F}$$

is $\dim X$ -regular (with respect to \mathcal{L}) for $n \geq n_0$. Hence $F_^n(\mathcal{E}) \otimes \mathcal{F} \otimes \mathcal{L}^{\dim X}$ is generated by global sections.*

Proof. Let $d = \dim X$ and write $\mathcal{L} = \mathcal{O}(1)$ (though \mathcal{L} may not be very ample). If $q > d$, then it is trivial that $H^q(F_*^n(\mathcal{E}) \otimes \mathcal{F}(d - q)) = 0$.

Now for $q = 1, \dots, d$, we have that $\mathcal{F}(d - q)$ is F -ample. So there exists n_0 such that $H^q(\mathcal{E} \otimes \mathcal{F}^{(p^n)}(p^n d - p^n q)) = 0$ for $n \geq n_0$.

Now since k is perfect, F is finite, so $H^q(F_*^n(\mathcal{E} \otimes \mathcal{F}^{(p^n)}(p^n d - p^n q))) = 0, q > 0$ [H1, Exercise III.4.1]. But since F is finite,

$$F_*^n(\mathcal{E} \otimes \mathcal{F}^{(p^n)}(p^n d - p^n q)) \cong F_*^n(\mathcal{E}) \otimes \mathcal{F}(d - q)$$

by [Ar, Lemma 5.7]. So we have shown the d -regularity we desired. \square

Finally, we need a certain presumably well-known exact sequence of locally free sheaves. Assuming X is integral, a proof was presented in [Ke, Lemma 4.4], so we will only indicate how to reduce to that proof.

Lemma 3.3. *Let X be a projective scheme of pure dimension d , smooth over a perfect field of characteristic $p > 0$. Let $F : X \rightarrow X$ be the absolute Frobenius morphism. For any $n > 0$, there is an exact sequence of coherent, locally free sheaves*

$$0 \rightarrow \mathcal{K} \rightarrow F_*^n \omega_X \rightarrow \omega_X \rightarrow 0$$

where $\omega_X = \wedge^d \Omega_{X/k}$ is the dualizing sheaf of X .

Proof. It is obviously a local question whether the cokernel of $\mathcal{O}_X \rightarrow F_*^n \mathcal{O}_X$ is locally free, so we may check this on the connected components of X . But since X is smooth, the connected components are the reduced, irreducible components [H1, Remark III.7.9.1]. Thus we have reduced to the integral scheme case, and we may proceed as in [Ke, Lemma 4.4]. Finally, the sheaf $\omega_X = \wedge^d \Omega_{X/k}$ is still the dualizing sheaf for a smooth scheme of pure dimension d [AK, I.4.6].

As an interesting alternative when k is algebraically closed, consider any open affine integral $U \subset X$. The injection $\mathcal{O}_U \rightarrow F_*^n \mathcal{O}_U$ is split [BK, Proposition 1.1.6]. Thus the cokernel of $\mathcal{O}_X \rightarrow F_*^n \mathcal{O}_X$ is locally free. The proof then proceeds as before. \square

We may now turn to our main theorem. The proof is now quite simple.

Theorem 3.4. *Let X be a projective scheme of pure dimension d , smooth over a field k . Let \mathcal{L} be an ample, globally generated line bundle. Let \mathcal{F} be an F -ample coherent sheaf. Then*

$$\omega_X \otimes \mathcal{L}^d \otimes \mathcal{F}$$

is generated by global sections.

Proof. By the reductions of Section 2, we may assume that k is perfect and of positive characteristic. We will have to allow X to be reducible, but X will be smooth of pure dimension d .

By Lemma 3.2, there exists n such that $F_*^n(\omega_X) \otimes \mathcal{L}^d \otimes \mathcal{F}$ is generated by global sections. But quotients of globally generated sheaves are globally generated, so tensoring the exact sequence of Lemma 3.3 by $\mathcal{L}^d \otimes \mathcal{F}$, we have the theorem. \square

We may now immediately prove Theorem 1.1 as a corollary.

Proof of Theorem 1.1. Since ample line bundles are exactly the F -ample line bundles [Ar, Lemma 2.4], we have Theorem 1.1(1) as an immediate corollary. We will need to use another technique of regularity to prove Theorem 1.1(2).

By the reductions of Section 2, we may assume that k is algebraically closed and of positive characteristic. Let $\mathcal{N} = \omega_X \otimes \mathcal{L}^{\dim X} \otimes \mathcal{H}$, which is free by Theorem 1.1(1).

Since k is algebraically closed and $\mathcal{N} \otimes \mathcal{L}$ is free, it is sufficient to show that $\mathfrak{m}_x \otimes \mathcal{N} \otimes \mathcal{L}$ is generated by global sections for any closed $x \in X$ [H3, p. 21]. To show this, we follow the method of [L, Example 1.8.22].

Let $\mathcal{E} = \bigoplus_{i=1}^{\dim X} \mathcal{L}$. Since \mathcal{L} is generated by global sections, \mathcal{E} has a section which vanishes on a zero-dimensional subscheme Z with $x \in Z$. Let \mathcal{I}_Z be the ideal sheaf of Z . Now \mathcal{I}_Z has a resolution derived from the Koszul complex of Z [L, Example 1.8.18, Appendix B.2]:

$$0 \rightarrow \wedge^d \mathcal{E}^* \rightarrow \cdots \rightarrow \wedge^2 \mathcal{E}^* \rightarrow \mathcal{E}^* \rightarrow \mathcal{I}_Z \rightarrow 0.$$

The exterior power $\wedge^q \mathcal{E}^*$ is a direct sum of \mathcal{L}^{-q} . By Lemma 3.2, there exists n such that $F_*^n(\omega_X) \otimes \mathcal{H}$ is $\dim X$ -regular. So we can tensor the Koszul complex with the locally free sheaf $F_*^n(\omega_X) \otimes \mathcal{L}^{\dim X+1} \otimes \mathcal{H}$, chase through long exact sequences, and find that $\mathcal{I}_Z \otimes F_*^n(\omega_X) \otimes \mathcal{L}^{\dim X+1} \otimes \mathcal{H}$ is 0-regular.

Any sheaf with 0-dimensional support is trivially 0-regular. So from the exact sequence

$$0 \rightarrow \mathcal{I}_Z \rightarrow \mathfrak{m}_x \rightarrow \mathfrak{m}_x/\mathcal{I}_Z \rightarrow 0$$

we get that $\mathfrak{m}_x \otimes F_*^n(\omega_X) \otimes \mathcal{L}^{\dim X+1} \otimes \mathcal{H}$ is also 0-regular and thus generated by global sections. The quotient sheaf $\mathfrak{m}_x \otimes \mathcal{N} \otimes \mathcal{L}$ is then generated by global sections and hence $\mathcal{N} \otimes \mathcal{L}$ is very ample. \square

4. FUJITA'S CONJECTURE FOR p -AMPLE SHEAVES

Another possible definition for “ampleness” of a coherent sheaf is p -ampleness. While previously only defined for vector bundles in characteristic p , the definition easily extends.

Definition 4.1. Let X be a projective scheme over a field k , and let \mathcal{F} be a coherent sheaf. If $\text{char } k = p > 0$ and F is the absolute Frobenius morphism, then define $\mathcal{F}^{(p^n)} = F^{*n} \mathcal{F}$. The sheaf \mathcal{F} is p -ample if for any locally free coherent sheaf \mathcal{E} , there exists n_0 such that

$$\mathcal{E} \otimes \mathcal{F}^{(p^n)}$$

is generated by global sections for $n \geq n_0$. If $\text{char } k = 0$, then \mathcal{F} is p -ample if and only if \mathcal{F}_s is p -ample for all closed fibers on some arithmetic thickening.

We chose the definition above because it immediately follows that an F -ample coherent sheaf is p -ample. To see this, if \mathcal{F} is F -ample, one has that $\mathcal{E} \otimes \mathcal{F}^{(p^n)}$ is 0-regular, and hence globally generated, for $n \gg 0$. However, we will see that we could have allowed \mathcal{E} to be any coherent sheaf.

Also, note that, in general, F -ample is a stronger condition than p -ample. Specifically, on \mathbb{P}^n , $n \geq 2$, the tangent bundle is p -ample but not F -ample [Ar, Example 5.9].

We now can prove another variant of Fujita's Conjecture.

Theorem 4.2. *Let X be a projective scheme of pure dimension d , smooth over a field k . Let \mathcal{L} be an ample, globally generated line bundle. Let \mathcal{F} be an F -ample coherent sheaf and let \mathcal{P} be a p -ample coherent sheaf. Then*

$$\omega_X \otimes \mathcal{L}^d \otimes \mathcal{F} \otimes \mathcal{P}$$

is generated by global sections. In particular, $\omega_X \otimes \mathcal{L}^{d+1} \otimes \mathcal{P}$ is generated by global sections.

Proof. By the reductions of Section 2, we may assume that k is perfect and of positive characteristic.

Since \mathcal{P} is p -ample, there exists n_0 such that $\omega_X \otimes \mathcal{P}^{(p^n)}$ is generated by global sections for $n \geq n_0$. We can apply the exact functor F_*^n to the surjection $\oplus \mathcal{O}_X \rightarrow \omega_X \otimes \mathcal{P}^{(p^n)}$. By the projection formula for finite morphisms [Ar, Lemma 5.7], there is a surjective homomorphism

$$(4.3) \quad \oplus F_*^n(\mathcal{O}_X) \rightarrow F_*^n(\omega_X) \otimes \mathcal{P} \rightarrow 0$$

for $n \geq n_0$.

Now by Lemma 3.2, $F_*^n(\mathcal{O}_X) \otimes \mathcal{L}^{\dim X} \otimes \mathcal{F}$ is 0-regular with respect to \mathcal{L} , and hence is generated by global sections, for $n \gg 0$. From Lemma 3.3 and the surjection (4.3), the quotient sheaf $\omega_X \otimes \mathcal{L}^{\dim X} \otimes \mathcal{F} \otimes \mathcal{P}$ is also generated by global sections. \square

As noted in the introduction, Theorem 1.1 was already known for $k = \mathbb{C}$, via the Kodaira Vanishing Theorem and Castelnuovo-Mumford regularity. There is a Kodaira Vanishing Theorem for F -ample sheaves (that is, for F -ample \mathcal{F} , $H^q(\omega_X \otimes \mathcal{F}) = 0, q > 0$ [Ar, Corollary 8.6]). Thus in characteristic 0, Theorem 3.4 could be proven via this “Kodaira Vanishing–regularity” method. However, [Ke, Remark 7.4] implies that there cannot be a Kodaira Vanishing Theorem for p -ample sheaves, so that method is not an option for Theorem 4.2.

We now turn to Theorem 1.2. Let $\{\mathcal{F}_n\}$ be a sequence of coherent sheaves. In [Ke], the author thoroughly examined under what conditions $\mathcal{E} \otimes \mathcal{F}_n$ had vanishing higher cohomology for $n \gg 0$. It was found that this vanishing occurs if and only if for any invertible sheaf \mathcal{H} , we have that $\mathcal{H} \otimes \mathcal{F}_n$ is F -ample for $n \gg 0$ [Ke, Theorem 1.3]. We now prove the global generation analogue.

Proof of Theorem 1.2. That (1) implies (2) is trivial. Assuming (2), let \mathcal{L} be an ample invertible sheaf and let \mathcal{H} be any invertible sheaf. For $n \gg 0$, we have that $\mathcal{L}^{-1} \otimes \mathcal{H} \otimes \mathcal{F}_n$ is generated by global sections. Thus $\mathcal{H} \otimes \mathcal{F}_n$ is a quotient of $\oplus \mathcal{L}$. It is easy to see that direct sums and quotients of p -ample sheaves are p -ample. Hence we have (3).

Now assume (3). Let \mathcal{G} be a coherent sheaf and let \mathcal{L} be an invertible sheaf, ample and generated by global sections. There exists m such that $\mathcal{G} \otimes \mathcal{L}^m$ is generated by global sections. By the assumption on $\{\mathcal{F}_n\}$, there exists n_0 such that $\omega_X^{-1} \otimes \mathcal{L}^{-m-\dim X-1} \otimes \mathcal{F}_n$ is p -ample for $n \geq n_0$. Then by Theorem 4.2,

$$\mathcal{L}^{-m} \otimes \mathcal{F}_n \cong (\omega_X \otimes \mathcal{L}^{\dim X+1}) \otimes (\omega_X^{-1} \otimes \mathcal{L}^{-m-\dim X-1} \otimes \mathcal{F}_n)$$

is generated by global sections for $n \geq n_0$. Thus $\mathcal{G} \otimes \mathcal{F}_n$ is generated by global sections, as it is a tensor product of globally generated sheaves. \square

Remark 4.4. The fact that 1.2(2) implies 1.2(1) shows that we could have allowed the \mathcal{E} in Definition 4.1 to be any coherent sheaf. On the other hand, the analogue for F -ample sheaves is not known to be true, unless the F -ample sheaf only fails to be locally free on a subscheme of dimension ≤ 2 [Ke, Lemma 3.11].

Remark 4.5. Instead of just taking a sequence of coherent sheaves, we could have indexed by a *filter* (that is, a partially ordered set such that for any α, β , there exist γ with $\alpha < \gamma, \beta < \gamma$). Thus, Theorem 1.2 answers [Ke, Question 7.5] in the

affirmative, at least when X is smooth. Based on the results of [Ke, Theorem 7.2], we conjecture that Theorem 1.2 will remain true for any projective X , at least when the \mathcal{F}_n are locally free. However, we do not see a way to reduce to the smooth case, as in [Ke, Section 5].

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